Dispersion reduction schemes for the 3D acoustic wave equation

Student: Mia Escobar, Advisor: Dr. Yajun An

Key Concepts

PDE Inverse Problems: Inverse problems involve numerically reconstructing some aspect of a partial differential equation (PDE) based on known solutions of the equation. A common method is to use Finite Difference (FD) schemes as they are flexible, robust, and easy to implement.

Numerical Dispersion: Numerical Dispersion occurs when using FD schemes to approximate PDEs due to computing a continuous equation in a discrete manor. This is especially prevalent at high frequencies.

3D Acoustic wave equation:

$$c^2(\partial_x^2 + \partial_y^2 + \partial_z^2) - \partial_t^2 u = 0$$

Wave Field = $u(x, y, z)$

Wave Speed= c

Exact dispersion:

$$\omega^2 = c^2(k^2\cos^2\theta\cos^2\phi + k^2\cos^2\theta\sin^2\phi + k^2\sin^2\theta)$$

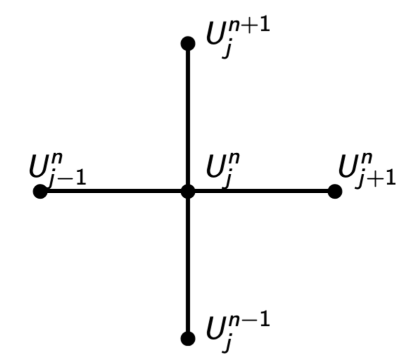
ω= frequency
k= wavelength
ϕ= azimuth angle
θ= plane wave propagation

Methods:

Finite Difference Schemes:

Finite Difference Schemes use derivatives and a nearby point to approximate a value.

We utilize a finite difference operator where we can change the number of points (M) used. Generally, higher M is more accurate.



Interpolation:

Interpolation at is achieved by replacing the last few accuracy conditions by exact dispersion.

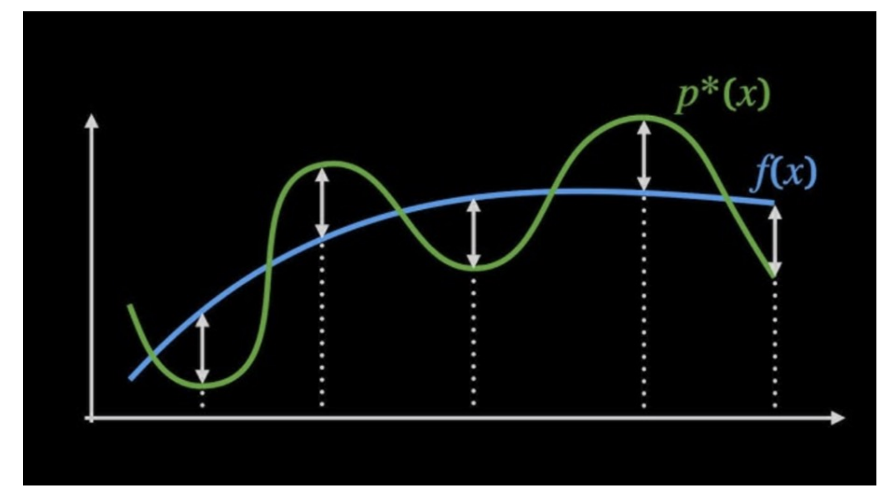
$$\begin{bmatrix} f_{1,1,\theta,\phi} & f_{1,2,\theta,\phi} & \cdots & f_{1,j,\theta,\phi} & \cdots & f_{1,M,\theta,\phi} \\ f_{2,1,\theta,\phi} & f_{2,2,\theta,\phi} & \cdots & f_{2,j,\theta,\phi} & \cdots & f_{2,M,\theta,\phi} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{i,1,\theta,\phi} & f_{i,2,\theta,\phi} & \cdots & f_{i,j,\theta,\phi} & \cdots & f_{i,M,\theta,\phi} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{M,1,\theta,\phi} & f_{M,2,\theta,\phi} & \cdots & f_{M,j,\theta,\phi} & \cdots & f_{M,M,\theta,\phi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_j \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} -(\gamma)^2 \\ -(\gamma)^4 \\ \vdots \\ -(\gamma)^{2j} \\ \vdots \\ -(\gamma)^{2M} \end{bmatrix}$$

$$\begin{bmatrix} f_{1,1,\theta,\phi} & f_{1,2,\theta,\phi} & \cdots & f_{1,j,\theta,\phi} & \cdots & f_{1,M,\theta,\phi} \\ f_{2,1,\theta,\phi} & f_{2,2,\theta,\phi} & \cdots & f_{2,j,\theta,\phi} & \cdots & f_{2,M,\theta,\phi} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{K_j,1,\theta,\phi} & g_{K_j,2,\theta,\phi} & \cdots & g_{K_j,j,\theta,\phi} & \cdots & g_{K_j,M,\theta,\phi} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{K_M,1,\theta,\phi} & g_{K_M,2,\theta,\phi} & \cdots & g_{K_M,j,\theta,\phi} & \cdots & g_{K_M,M,\theta,\phi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_j \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} -(\gamma)^2 \\ -(\gamma)^4 \\ \vdots \\ -\cos(K_j) \\ \vdots \\ -\cos(K_M) \end{bmatrix}$$

The Remez (Exchange) Algorithm:

To guarantee accurate dispersion for all wavenumbers/ propagation angles of g, we utilize the Remez Exchange Algorithm, which utilizes minimax approximation.

Minimax approximation can be used for functions in a Haar space. It takes the difference between the maxima and shifts the function f(x) to the optimal position that minimizes error.



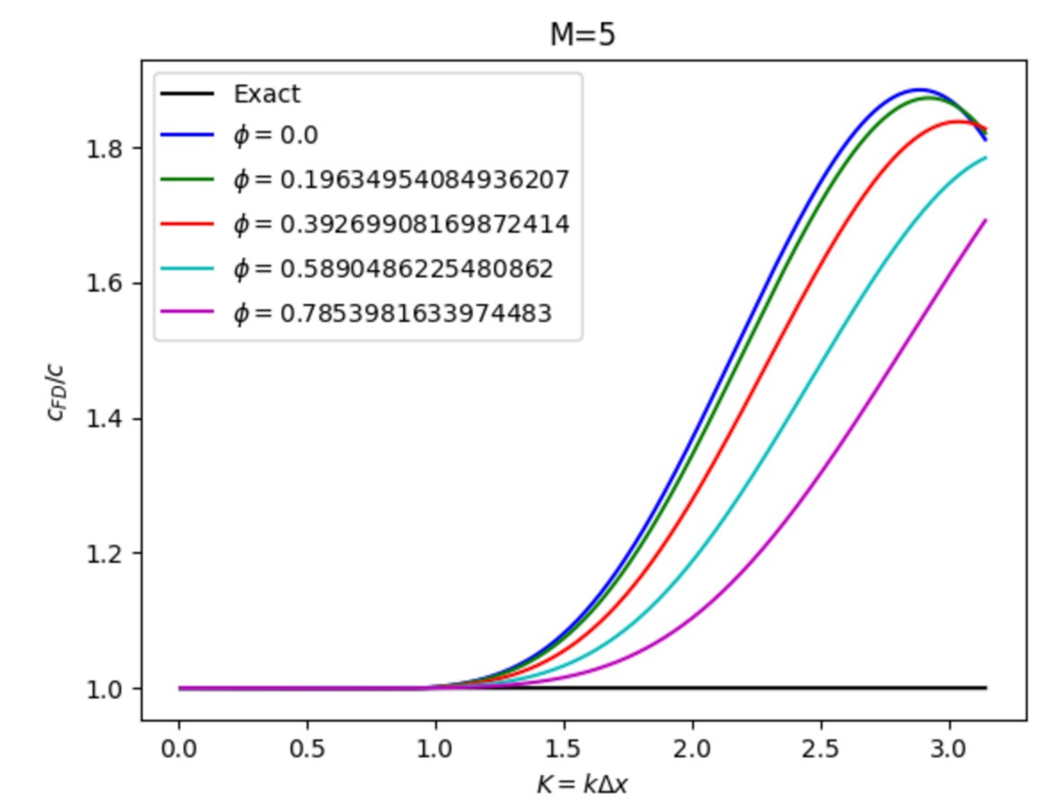
1. Pick a set of reference points $\{K_i\}_{i=1}^{M+1-l} \in (0, \pi]$. Denote $h = f(K_1) - p(K_1, \phi_{preset})$, h is currently unknown and will be solved next step.
2. Find the first polynomial $p(K, \phi_{preset}) = \sum_{m=0}^{M} c_m [\cos(mK\cos(\phi_{preset})) + \cos(mK\sin(\phi_{preset}))]$ and h by solving

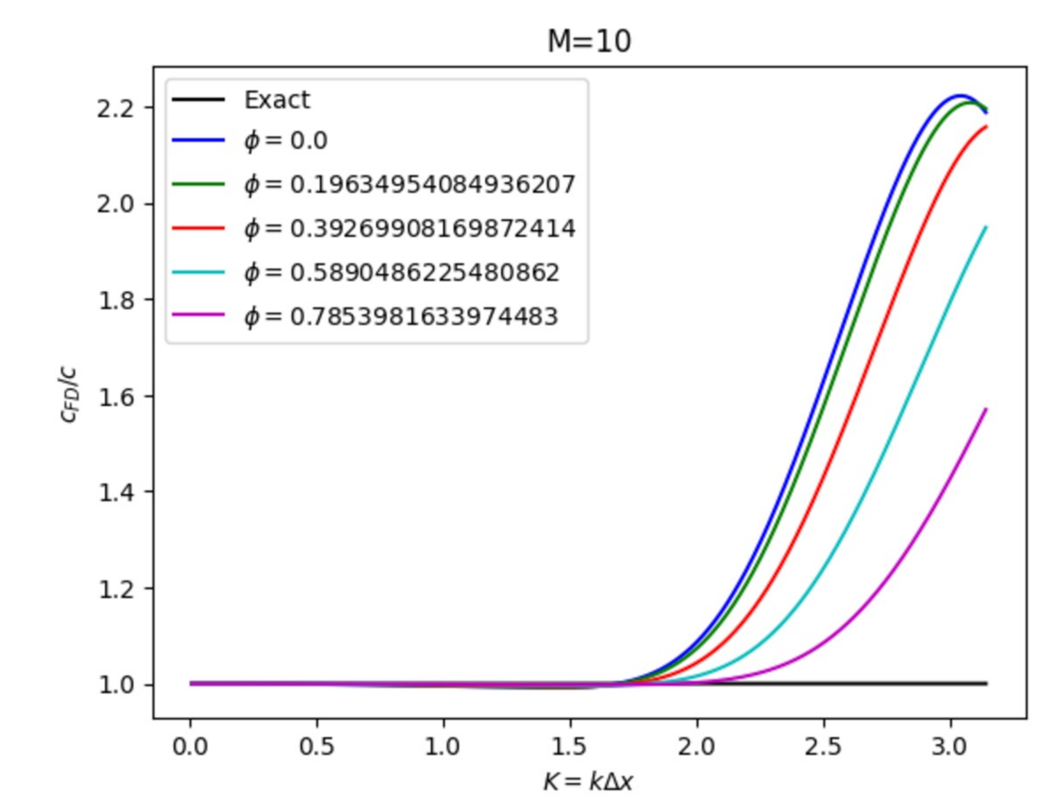
   $p(0, \phi_0) = 1, \quad j = 0$

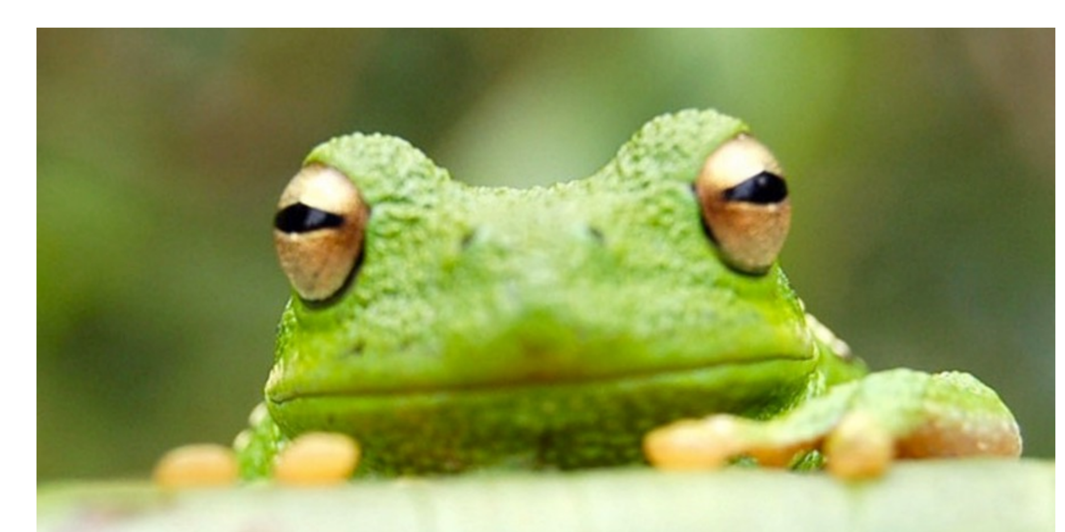
   $p^{(2j)}(0, \phi_0) = (-1)^j \gamma^{2j}, \quad j = 1, \ldots, l$

   $p(K_i, \phi_{preset}) + (-1)^{i-1} h = f(K_i, \phi_{preset}), \quad i = 1, \ldots, M + 1 - l.$
3. Compute the error function $e(K) = f(K) - p(K, \phi_{preset})$. Find η, a point where $|f(\eta) - p(\eta, \phi_{preset})| = ||f - p||$, and denote $\delta = |f(\eta) - p(\eta, \phi_{preset})| - |h|$.
   - If the difference δ is within the preset accuracy, stop and take p to be the best approximation.
   - If not, replace a point from $\{K_i\}_{i=1}^{M+1-l}$ with η, denote the new set of reference points by $\{K_i^+\}_{i=1}^{M+1-l}$. Let $h = f(K_1^+) - p(K_1^+, \phi_{preset})$.
4. Repeat above until the difference is below the preset error bound.

Results







RM Alford, KR Kelly, and D Mt Boore. Accuracy of finite-difference modeling of the acoustic wave equation. Geophysics, 39(6):834–842, 1974.
Yajun An. Uniform dispersion reduction schemes for the one dimensional wave equation in isotropic media. Journal of Computational Physics, 341:13–21, 2017.
Yajun An. Time-space domain dispersion reduction schemes in the uniform norm for the 2d acoustic wave equation. Journal of Computational Physics, 445:110589, 2021.
Bezalel Finkelstein and Raphael Kastner. Finite difference time domain dispersion reduction schemes. Journal of Computational Physics, 221(1):422–438, 2007.
Bezalel Finkelstein and Raphael Kastner. A comprehensive new methodology for formulating fdtd schemes with controlled order of accuracy and dispersion. IEEE Transactions on Antennas and Propagation, 56(11):3516–3525, 2008.
Dan Kosloff, Reynam Pestana, and Hillel Tal-Ezer. Numerical solution of the constant density acoustic wave equation by implicit spatial derivative operators. In SEG Technical Program Expanded Abstracts 2008, pages 2057–2061. Society of Exploration Geophysicists, 2008.
Dan Kosloff, Reynam C Pestana, and Hillel Tal-Ezer. Acoustic and elastic numerical wave simulations by recursive spatial derivative operators. Geophysics, 75(6):T167–T174, 2010.
E Kimchi and N Richter-Dyn. Best uniform approximation with hermite-birkhoff interpolatory side conditions. Journal of Approximation Theory, 15(2):85–100, 1975.
E Kimchi and N Richter-Dyn. Properties of best approximation with interpolatory and restricted range side conditions. Journal of Approximation Theory, 15(2):101–115, 1975.
Samuel Karlin and William J Studden. Tchebycheff systems: With applications in analysis and statistics. (No Title), 1966.
Kenneth R Kelly, Ronald W Ward, Sven Treitel, and Richard M Alford. Synthetic seismograms: a finite-difference approach. Geophysics, 41(1):2–27, 1976.
Alan R Levander. Fourth-order finite-difference p-sv seismograms. Geophysics, 53(11):1425–1436, 1988.
Keran Li and Wenyuan Liao. An efficient and high accuracy finite-difference scheme for the acoustic wave equation in 3d heterogeneous media. Journal of Computational Science, 40:101063, 2020.
Yang Liu and Mrinal K. Sen. A new time–space domain high-order finite-difference method for the acoustic wave equation. Journal of Computational Physics, 228(23):8779 8806, 2009.
Yang Liu and Mrinal K Sen. Time–space domain dispersion-relation-based finite-difference method with arbitrary even-order accuracy for the 2d acoustic wave equation. Journal of Computational Physics, 232(1):327–345, 2013.
Oded Ovadia, Adar Kahana, and Eli Turkel. A convolutional dispersion relation preserving scheme for the acoustic wave equation. Applied Mathematics and Computation, 461:128317, 2024.
Oded Ovadia, Adar Kahana, Eli Turkel, and Shai Dekel. Beyond the courant-friedrichs lewy condition: Numerical methods for the wave problem using deep learning. Journal of
Computational Physics, 442:110493, 2021.
Michael James David Powell. Approximation theory and methods. Cambridge university press, 1996.
Jean Virieux. Sh-wave propagation in heterogeneous media: Velocity-stress finite-difference method. Geophysics, 49(11):1933–1942, 1984.
Jean Virieux. P-sv wave propagation in heterogeneous media: Velocity-stress finite- difference method. Geophysics, 51(4):889–901, 1986.
Shumin Wang and F.L. Teixeira. Dispersion-relation-preserving fdtd algorithms for large scale three-dimensional problems. IEEE Transactions on Antennas and Propagation, 51(8):1818–1828, 2003.
Shumin Wang and F.L. Teixeira. Three-dimensional dispersion-relation-preserving finite- difference time-domain schemes. In IEEE Antennas and Propagation Society International Symposium. Digest. Held in conjunction with: USNC/CNC/URSI North American Radio Sci. Meeting (Cat. No.03CH37450), volume 4, pages 332–335 vol.4, 2003.