Fun with Pfaffians

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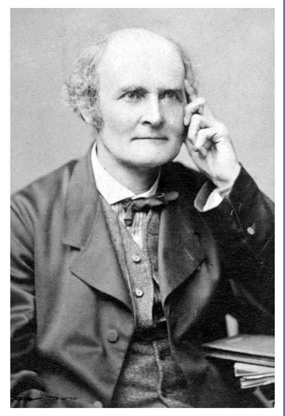
TMATH 498 Independent Reading | Jennifer Quinn, Ph.D

History

Pfaffians are a matrix function introduced in 1815. Arthur Cayley proved a connection between the Pfaffian and determinant of a matrix. Specifically, for skew-symmetric matrices, the determinant is equal to the Pfaffian squared.



A. Cayley

Definitions

The determinant:
Given an $n \times n$ matrix A, the determinant of $A = \langle a_{i,j} \rangle$ is

$$\det(A) = \sum_{\sigma \epsilon S_n} \operatorname{sgn}(\sigma)\ a_{1,\sigma_1} a_{2\sigma_2} \dots a_{n,\sigma_n}$$

with S_n being all the permutations of $[n] = \{1,2,\dots,n\}$.

The Pfaffian:
Given a $2n \times 2n$ skew-symmetric matrix $A = \langle a_{ij} \rangle$, the Pfaffian is

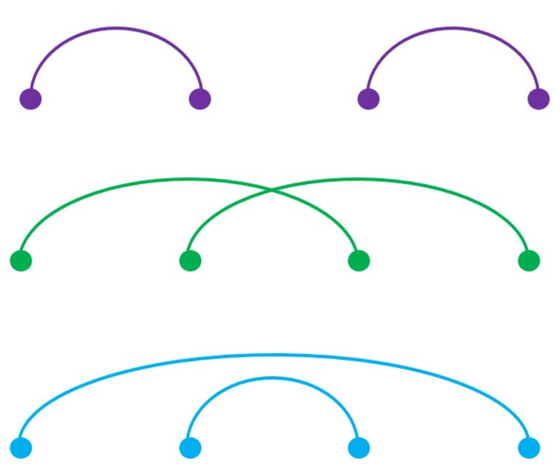
$$Pf(A) = \sum_{\rho} \operatorname{sgn}(\rho)\ a_{\rho_1,\rho_2} \dots a_{\rho_{2n-1},\rho_{2n}}$$

where ρ is a matching and $\operatorname{sgn}(\rho)$ is -1 raised to the number of chord crossings in ρ.

Example

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ -2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 3 \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

Of $\{1,2,3,4\}$, there are three possible matchings:



These correspond with the following elements of the matrix:

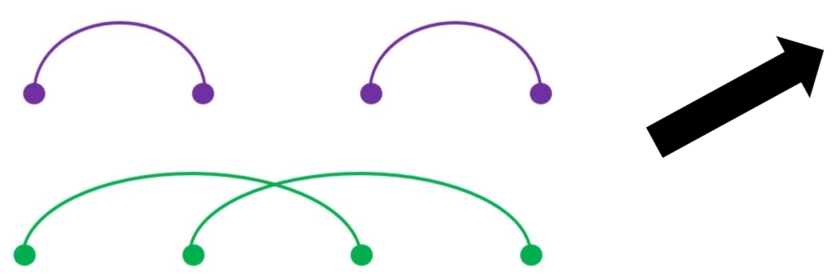
$$\begin{bmatrix} 0 & \boxed{2} & \boxed{1} & \boxed{1} \\ -2 & 0 & \boxed{1} & \boxed{0} \\ -1 & -1 & 0 & \boxed{3} \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

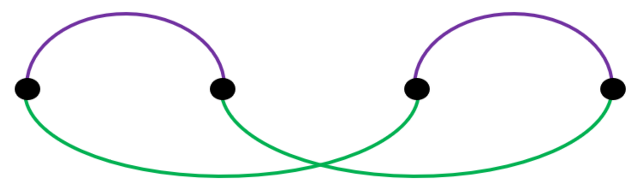
Contributions to the sum:
Purple: $1 \cdot (2 \cdot 3) = 6$
Green: $-1 \cdot (1 \cdot 0) = 0$
Blue: $1 \cdot (1 \cdot 1) = 1$
Total: $6 + 0 + 1 = 7$.

Thus $Pf(A) = 7$. The determinant is the Pfaffian squared, so $\det(A) = 49$

Matchings and Permutations

Let's combine two matchings:

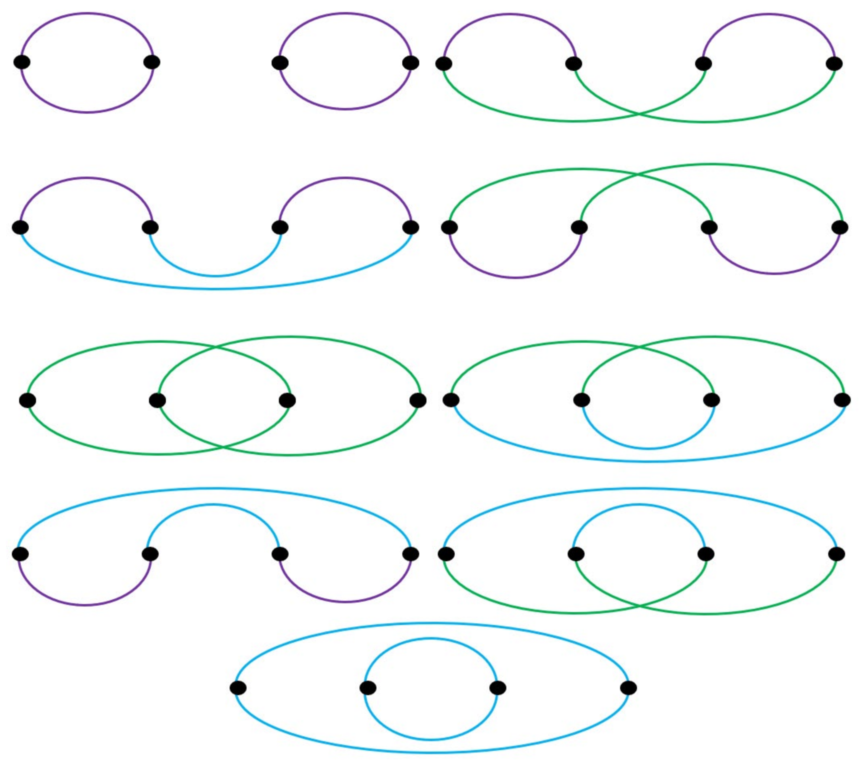




- We created a permutation of even length cycles!
- Cyclic notation: $(1,2,4,3)$.
- Contribution of 0 to $\det(A)$.

$$\begin{bmatrix} 0 & \boxed{2} & 1 & 1 \\ -2 & 0 & 1 & \boxed{0} \\ \boxed{-1} & -1 & 0 & 3 \\ -1 & 0 & \boxed{-3} & 0 \end{bmatrix}$$

Let's make more!



There are $3^2 = 9$ such even permutations. This is not a coincidence!

- These permutations contribute 49 to $\det(A)$.
- All other permutations either contribute zero or cancel out with one another.

Odd-order Skew-symmetric Matrices

Given an $n \times n$ skew-symmetric matrix A where n is odd, $\det(A) = Pf(A)^2 = 0$.

- Reverse the first odd-length cycle to create σ^*.
- Same magnitude, different sign, everything cancels!

$\sigma = (123)(45)$

$$\begin{bmatrix} 0 & \boxed{2} & -3 & 0 & 1 \\ -2 & 0 & \boxed{-8} & 7 & 2 \\ \boxed{3} & 8 & 0 & 5 & 2 \\ 0 & -7 & -5 & 0 & \boxed{1} \\ -1 & -2 & -2 & \boxed{-1} & 0 \end{bmatrix}$$

$\sigma^* = (132)(45)$

$$\begin{bmatrix} 0 & 2 & \boxed{-3} & 0 & 1 \\ \boxed{-2} & 0 & -8 & 7 & 2 \\ 3 & \boxed{8} & 0 & 5 & 2 \\ 0 & -7 & -5 & 0 & \boxed{1} \\ -1 & -2 & -2 & \boxed{-1} & 0 \end{bmatrix}$$

Pfaffian Exploration

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

What's the Pfaffian of A? Look at the matchings:

$\rho_1 = \{1,2\}\{3,4\}$
$\rho_2 = \{1,3\}\{2,4\}$
$\rho_3 = \{1,4\}\{2,3\}$

ρ_1 contributes 1,
ρ_2 contributes -1,
ρ_3 contributes 1.
The Pfaffian is 1!

Question: Can we generalize this?

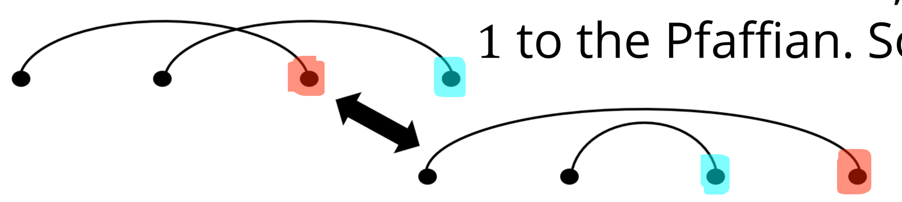
For a $2n \times 2n$ skew-symmetric matrix whose upper triangular is all ones, what is the Pfaffian?

$$A = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 0 \end{bmatrix}$$

$$Pf(A) = ?$$

For $\rho = \{\rho_1, \rho_2\}, \dots, \{\rho_{n-1}, \rho_n\}$, find the first chord whose length ≥ 2 and do a swap with the subsequent chord. New matching is ρ'.

Example with ρ_2:
$\{1,3\}\{2,4\}$
Swap!
$\{1,4\}\{2,3\}$



Every ρ has a ρ' except the consecutive chord, which contributes 1 to the Pfaffian. So $Pf(A) = 1!$

$\rho_2{}' = \rho_3$ and $\rho_3{}' = \rho_2$

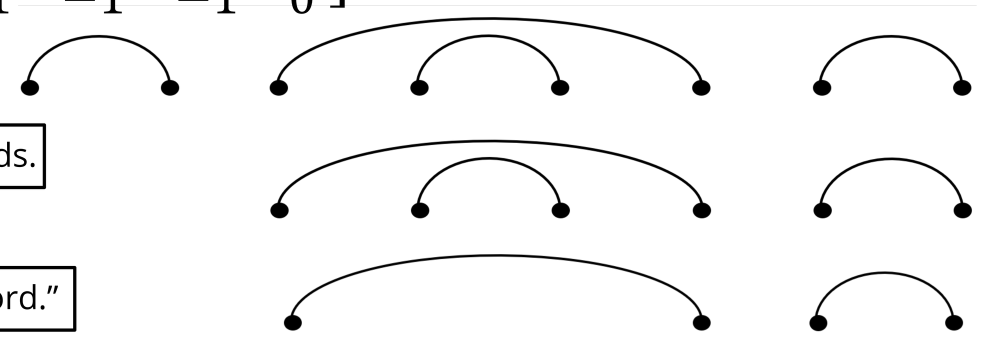
Adding Zeros

Question: Can we take the prior example and change a 1 to a 0?

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & \boxed{0} & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Any matching ρ with the "zero chord" now contributes zero to the Pfaffian.

What if ρ has the "zero chord" but ρ' doesn't?



1. Remove the leading consecutive chords.
2. Remove the "zero chord."

- We can make the same correspondence as the prior example.
- All the "reduced" matchings cancel except the consecutive one.
- $Pf(A)$ is either 0 or 2 depending on whether removing the "zero-chord" changed the sign.

What does this look like?

Put a 0 here and $Pf(A) =$
0 2

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

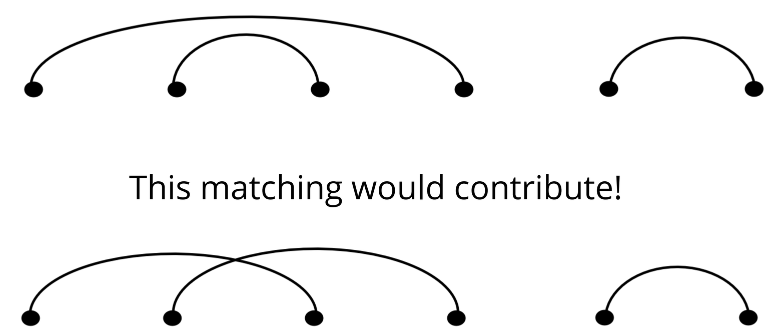
Adding More Zeros

A $2n \times 2n$ skew-symmetric matrix with whose upper triangular consists of alternating diagonals of 1s and 0s, where the super-diagonal is all 1s, will be 2^{n-1}.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

The Pfaffian of this matrix is $2^{3-2} = 4$.

Even chords zero out the matching, so only matchings of all odd chords contribute.



- We can create a correspondence between odd-chord matchings that cancels most of them out.
- Remaining matchings can be mapped to even-order subsets of the set with 2^n elements

References

Cameron, Quinn. Pfaffians are Pfine. Math Magazine, to appear.

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Recognition

- Thanks to Dr. Jennifer Quinn for supervising and supporting this exploration.
- Thanks to the entire math department, students and faculty, for providing such a supportive environment for mathematical growth.