Fun with Pfaffians

Kaioke Begay & Dr. Jennifer Quinn

Pfaffians are matrix functions which were first introduced in 1815 by Johann Friedrich Pfaff. Arthur Cayley would later give Pfaffian's their name and prove their connection to determinants. Specifically, the determinant of a matrix is equal to the Pfaffian squared. It is important to note that Pfaffians only apply to skew-symmetric matrices, with a square matrix A being defined as skew-symmetric if $A = -A^T$, where A^T denotes the transpose of A. The Pfaffian of a matrix can be expressed as the sum of all the possible matchings of the matrix multiplied by the sign of the matching. Here Pfaffians are introduced, and the Pfaffians of several different types of skew-symmetric matrices are calculated. It can be shown that the Pfaffian of an $m \times m$ skew-symmetric is zero when m is odd; only a $2n \times 2n$ skew-symmetric matrix would give a non-zero Pfaffian. For a $2n \times 2n$ matrix with an upper diagonal of all ones, the Pfaffian is 1, which can be shown by creating a correspondence between matchings of opposite signs. By introducing a zero into the upper diagonal, the Pfaffian will become either 0 or 2 depending on the location of the introduced zero.