Teaching and Learning Center
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Induction Introduction

Mathematical induction is a method of proof used to show that a statement, usually called $P(n)$, is true for all integers greater than or equal to specific integer called the base case.

A proof by induction is usually completed in three steps:

Basis Step: Identify $P(n)$ and establish a base case. $P(n)$ is just the statement we are trying to prove. The base case is the lowest integer for which $P(n)$ could be true. For example, if the problem says "show $P(n)$ is true for all integers $n \geq 3$," then the base case will be $n = 3$ and we would have to show that $P(3)$ is true. If the problem says "show $P(n)$ is true for all positive integers," then the base case would be $n = 1$ because that is the smallest positive integer. We would then show that $P(1)$ is true.

Inductive Hypothesis: Assume the inductive hypothesis. This step is just the assumption that $P(n)$ is true up to a certain value $n = k$. State that $P(k)$ is true.

Inductive Step: Show that $P(k + 1)$ is true using the inductive hypothesis. We want $P(k + 1)$ to be true so we should write out exactly how that would look. Start with the inductive hypothesis and manipulate the equation or inequality to get the statement we want (this is usually the hard part). Once we show that $P(k + 1)$ is true, we are done with the proof. We've shown that if $P(k)$ is true, then $P(k + 1)$ is also true. Since we established a base case at a certain number, we know that $P(n)$ is true for all integers beyond that number. For example, if we established in the basis step that $P(1)$ is true, then the inductive step showed that $P(2)$ must be true. If $P(2)$ is true, then $P(3)$ is true and so on. We've proved conclusively that the statement $P(n)$ is true for all integers greater than or equal to the base case.

Some Examples

Example 1 Show that if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Basis Step: $P(n)$ is the statement $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We're trying to prove this statement for all positive integers so the base case should be $n = 1$. $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$. Is this true? $1 = \frac{1(1+1)}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1$. Yes it is. We have shown $P(1)$ is true thus establishing a base case.

Inductive Hypothesis: Assume $P(n)$ is true up to $n = k$. $P(k)$ is true therefore the statement $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true.

Inductive Step: Show $P(k+1)$ is true using the assumption that $P(k)$ true. Since we want $P(k+1)$ to be true, the statement we want to ultimately arrive at is

$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$.

Start with $P(k)$ since we know that is true.

$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)+2(k+1)}{2}$

$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$ which is exactly what we wanted to show. We have shown $P(k+1)$ is true if $P(k)$ is true. The proof is now concluded. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$. ■

Example 2 Show that $1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Basis Step: $P(n)$ is the statement $1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. Since we are proving this statement for all nonnegative integers, the base case should be $n = 0$ because that is the smallest nonnegative integer. $P(0)$ is the statement $2^0 = 2^{0+1} - 1$. Is this true?

$2^0 = 2^{0+1} - 1$

$1 = 2^1 - 1$

$1 = 2 - 1 = 1$. Yes it is. We have shown $P(0)$ is true thus establishing a base case.

Inductive Hypothesis: Assume $P(n)$ is true up to $n = k$. $P(k)$ is true therefore the statement $1 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ is true.

Inductive Step: Show $P(k+1)$ is true using the assumption that $P(k)$ true. Since we want $P(k+1)$ to be true, the statement we want to ultimately arrive at is

$1 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1.$

Start with $P(k)$ since we know that is true.

$1 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$

$1 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

$1 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2 * 2^{k+1} - 1$

$1 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$ which is exactly what we wanted to show.

We have shown $P(k+1)$ is true if $P(k)$ is true. The proof is now concluded.

$1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ■

We can use induction to prove inequalities.

Example 3 Show that $3^n < n!$ if n is an integer greater than 6.

Basis Step: $P(n)$ is the statement $3^n < n!$ The base case should be $n = 7$ because it is the smallest integer greater than 6. $P(7)$ is the statement $3^7 < 7!$ Is this true?

$3^7 < 7!$

$2187 < 5040$. Yes it is. We have shown $P(7)$ is true thus establishing a base case.

Inductive Hypothesis: Assume $P(n)$ is true up to $n = k$. $P(k)$ is true therefore the statement $3^k < k!$ is true.

Inductive Step: Show $P(k+1)$ is true using the assumption that $P(k)$ is true. Since we want $P(k+1)$ to be true, the statement we want to ultimately arrive at is

$3^{k+1} < (k+1)!$

Start with $P(k)$ since we know that is true.

$3^k < k!$

$3 * 3^k < 3 * k!$

$3^{k+1} < 3 * k! < (k+1) * k!$ since $3 < k + 1$ because $k > 6$

$3^{k+1} < (k+1) * k!$

$3^{k+1} < (k+1)!$ which is exactly what we wanted to show.

We have shown $P(k+1)$ is true if $P(k)$ is true. The proof is now concluded.

$3^n < n!$ for all $n > 6$. ■

We can use induction to prove divisibility results.

Example 4 Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Basis Step: $P(n)$ is the statement $2|(n^2 + n)$. The base case should be $n = 1$ because it is the smallest positive integer. $P(1)$ is the statement $2|(1^2 + 1)$. Is this true?

$2|(1^2 + 1)$

$2|(1 + 1)$

$2|2$. Yes it is. We have shown $P(1)$ is true thus establishing a base case.

Inductive Hypothesis: Assume $P(n)$ is true up to $n = k$. $P(k)$ is true therefore the statement $2|(k^2 + k)$ is true.

Inductive Step: Show $P(k + 1)$ is true using the assumption that $P(k)$ is true. Since we want $P(k + 1)$ to be true, the statement we want to ultimately arrive at is

$2|((k + 1)^2 + k + 1)$.

Start with $P(k)$ since we know that is true.

$2|(k^2 + k)$

$2|(k^2 + k + 2k)$

$2|(k^2 + k + 2k + 2)$

$2|(k^2 + k + 2k + 1 + 1)$

$2|(k^2 + 2k + 1 + k + 1)$

$2|((k + 1)^2 + k + 1)$ which is exactly what we wanted to show.

We have shown $P(k + 1)$ is true if $P(k)$ is true. The proof is now concluded.

$2|(n^2 + n)$ for all $n \geq 1$. ∎